Time Interval Averaging: Theory, Problems, and Solutions

by David C. Chu

TIME INTERVAL AVERAGING is an easy and economical way to increase resolution in measuring repetitive time intervals. The idea is quite simple: the same interval is measured repeatedly and, given some degree of independence between measurements, the ±1 count quantization error in each measurement is statistically reduced if the average measurement value is used to estimate the interval.

There are many pitfalls in making measurements this way. My purpose here is to point these out and describe the approaches used in the 5345A to solve these problems.

The Folly of Direct Gating

In one all too obvious implementation of time interval averaging, the time interval repeatedly enables a gate through which the clock pulses are passed and counted. Unfortunately, when the gate turns on and off, partial clock pulses are generated and fed to the counting circuits. The average value obtained this way is as much a function of the response of the counter to partial pulses as it is to the width of the interval. Because this response is difficult to characterize reliably and even harder to control accurately, one can attach no meaningful significance to the average value so obtained. In general, for averaging over a large number of measurements, a biased value, which can differ significantly from the true time interval value, will be approached. The difference can be expressed as:

$$\text{partial pulse bias} = T_0 \frac{d}{100}(1 - 2r) \qquad (1)$$

where T_0 is the clock period, d is the duty cycle of the clock in percent, and r is the fraction of the full clock pulse below which the counter does not respond. The parameter r, which is always between 0 and 1, cannot be controlled precisely.

The use of a synchronizer in HP averaging counters eliminates the partial pulse problem by reducing the effective duty cycle (d) to zero.[1] The decades are fed full-width pulses under all conditions. The remainder of this article assumes the use of synchronizers.

Non-Averaging Because of Coherence

Another fundamental problem of time interval averaging occurs when the time intervals are repeated at a rate coherent with the clock frequency. For example, if the rate is a submultiple of the clock frequency, the occurrence of the time interval relative to the clock phase is the same for each measurement. Hence, all the measurements read exactly the same and no statistical averaging takes place. In this case, the quantizing error for a million measurements is no different from that for a single measurement.

Coherence, unfortunately, is not limited to submultiples. There are other rates at which only partial averaging takes place. For example, rates given by $f_0/(Q+½)$, where f_0 is the clock frequency and Q is a positive integer, give rise to two alternating discrete clock phases separated by $T_0/2$. Averaging over a large number of measurements is no better than averaging over two successive measurements.

In general, we can partition the coherent rates into classes. A "class-M" rate results in the time interval's occurring at M discrete phases of the clock. If one is interested in gaining resolution improvement by, say, a factor of 100 over the clock period, a class-M rate where M is under 100 would be unacceptable. A class-M rate is given by:

$$f_R(M) = \frac{f_0}{Q + \frac{L}{M}} \qquad M = 1,2,3... \qquad (2)$$

where $f_R(M)$ is a class-M rate, Q, L, and M are non-negative integers, and $L \leq M$. Furthermore, L and M are co-prime, that is, they have no common factors. For M = 1, the class of submultiples is generated.

These rates are very numerous. In fact, there is an infinite number of rates for each class.

Coherence Bandwidth

If one is interested in gaining resolution by a factor of N, how far must the time interval rate depart from one of the coherent rates f_R to assure proper averaging? Straightforward analysis of the phase of the intervals at these frequencies shows that for a class-M rate f_R, a departure of $\pm\Delta f_R$ from f_R can result in almost perfect averaging by the factor N, where Δf_R is given by:

$$\Delta f_R = \frac{f_R^2}{f_0 MN} \qquad (3)$$

The coherence bandwidth is $2\Delta f_R$.

In terms of fractional frequency stability $\Delta f_R/f_R$, the stability that would cause the non-averaging effect is:

$$\frac{\Delta f_R}{f_R} = \frac{f_R}{f_0 MN} \qquad (4)$$

Conversely, any stability worse than this destroys the coherence and allows the reduction of measurement quantization error by statistics.

The coherence bandwidth is large for high rates and low M, and non-averaging can often be observed without instrumentation. Submultiples (M=1), for example, give rise to measurements always equal to or close to whole clock periods, an easily discernible effect. Partial averaging is more subtle, and the experimenter is often led to accept a result which does not give an adequate interpolation factor.

Time Base Random Phase Modulation

Coherence between the time interval pulse train and the time base clock pulse train can be destroyed by introducing random phase modulation to either or both of the trains, allowing meaningful time interval averaging measurements to be made without regard to the time-interval rate. In the HP 5345A Counter, phase modulation is deliberately introduced into the clock pulse train when making time interval average and pulsed RF measurements. The modulation is random Gaussian noise band-limited to approximately 3 kHz.

The bias errors caused by the non-averaging effect are functions of many parameters, including the time interval rate, the fractional part of the time interval (obtained by subtracting all whole clock periods from the time interval), and the interval-to-clock phase relationship. However, the *worst case* bias error, E_B, can be expressed simply as a function of σ, the rms value of the phase modulation:

$$E_B = T_0\Big[\sum_{n=-\infty}^{\infty}\int_{n-1/4}^{n+1/4}\frac{1}{\sqrt{2\pi}\sigma}\exp\{\frac{x^2}{2\sigma^2}\}dx - 1/2\Big] \qquad (5)$$

where E_B is the non-averaging bias error and σ is the rms value of the phase modulation in units of 2π radians or periods of the time base clock.

Fig. 1 shows a plot of bias error E_B versus σ in radians, assuming a 2-ns clock period, and a time interval given by $2(Q+1/2)$ ns. The bias error decreases rapidly with increasing phase modulation. This indicates that the phase modulation should be large. However, as will be shown, another type of error demands just the opposite.

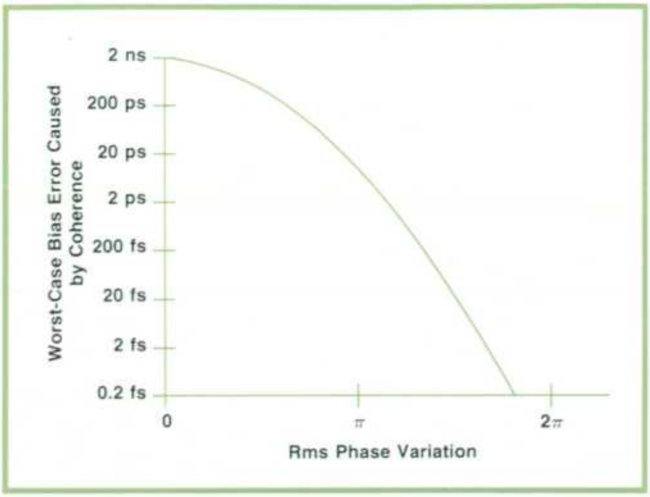

Fig. 1. *Coherence bias error in time interval averaging is reduced by increasing random phase modulation of the counter time base.*

Modulation Bandwidth

There are also two opposing requirements on the phase modulation bandwidth. With a large bandwidth, time intervals can arrive rapidly and still maintain relative independence between measurements. On the other hand, for very large modulation bandwidths and relatively long time intervals, there may be only limited correlation between the time base at the start edge and at the stop edge, and the measurement accuracy may be degraded by time base short-term instability.

Quantitatively, the relationship is as follows. For a given time interval τ seconds, modulation bandwidth f_c (Hz), and rms phase modulation σ (cycles), the rms error caused by time base uncertainty in a single measurement is given by E_{TB}, where

$$\text{rms value of } E_{TB} = T_0\sigma\big[2(1 - e^{-2\pi f_c|\tau|})\big]^{1/2} \qquad (6)$$

seconds. For a given time interval, this error is smaller for a smaller modulation bandwidth. Thus a small modulation bandwidth is desirable.

With averaging, this error is reduced by the factor $1\sqrt{N}$, where N is the number of *independent* measurements. Not all measurements averaged are independent if the time interval rate exceeds approximately twice the modulation bandwidth. In this latter respect, a large bandwidth is desirable because it makes full use of all measurements made at higher rates.

Notice that E_{TB} increases linearly with σ, the rms phase modulation, in contrast with the coherence error, which decreases with σ.

Normal ±1 Count Quantization Error

Even if the time interval rate is incoherent and time base uncertainty error is negligible (such as when measuring short time intervals), an error is expected in time interval average measurements because of the normal ±1 count quantization error in each measurement.* For a given time interval $\tau = T_0 (Q+F)$, where Q is an integer and F a proper fraction, this quantization error can take on only two fixed values, FT_0 and $(F-1)T_0$, with probabilities $(1-F)$ and F respectively. The mean of this distribution is zero, and the standard deviation (rms value) is $T_0[F(1-F)]^{1/2}$. The worst case occurs when F is 1/2, with the corresponding worst case rms quantization error of $T_0/2$. With proper averaging over N independent measurements, this rms error is reduced by the familiar factor $1/\sqrt{N}$.

Measurement Error Summary

Three types of errors in time interval averaging have been discussed. They are non-averaging bias error, time base short-term uncertainty error, and normal quantization error.

The first error is caused by coherence and can be as large as one whole clock count. It is independent of the number of measurements averaged. This error can be reduced to a negligible level according to Fig. 1 by randomly phase modulating the time base clock. In the 5345A Counter, random phase modulation of 0.8 cycle rms minimum allows meaningful time interval average measurements without regard to the time interval rate.

The second error is a result of time base uncertainty caused by the random phase modulation introduced. For measuring time intervals less than $7\mu s$ with the 5345A, this time base error is completely dominated by the quantization error, and is therefore negligible. For measuring time intervals much larger than $7\ \mu s$, the measurement deviation is increased by a factor of approximately 2.7 above that due to the normal ±1 count quantization error. This increase can be nullified by averaging more intervals.

The third type of error is a result of quantization to whole numbers in the counting process and is a function of many parameters. The worst case occurs when the time interval has a value half way between whole clock counts, giving an rms error $T_0/2$. Like time base uncertainty error, quantization error is reduced if averaged over N independent measurements by the factor $1/\sqrt{N}$.

There are other errors in time interval measurements, caused primarily by non-ideal circuit components. Examples are start-stop channel mismatch, trigger error, and plain old thermal noise. These errors are important, but have been excluded from this discussion, which is limited to those errors that are unique to the time interval averaging process.

*"±1 count" is actually a misnomer when synchronizers are used, because they make it impossible to have an error exactly equal to one count.

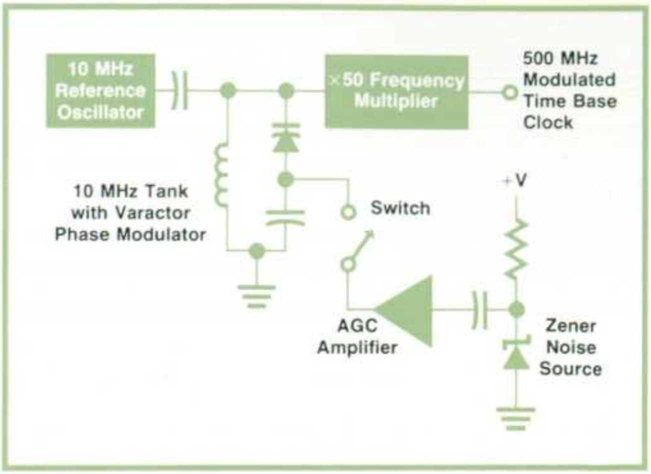

Fig. 2. *Implementation of time-base-clock random phase modulation in the HP 5345A. The switch is closed only for time interval average and pulsed RF measurements.*

In the 5345A Counter, the phase modulating signal is derived from noise generated by a zener diode. The noise is amplified and filtered before being used to modulate the phase of the clock at 10 MHz with an rms value of approximately 7°. The frequency multiplier chain effectively increases this value by a factor of 50 to about 350°. The noise voltage level is accurately controlled at all times by a feedback loop. A block diagram of the implementation scheme is shown in Fig. 2.

FM versus PM

There are two fundamental reasons why phase

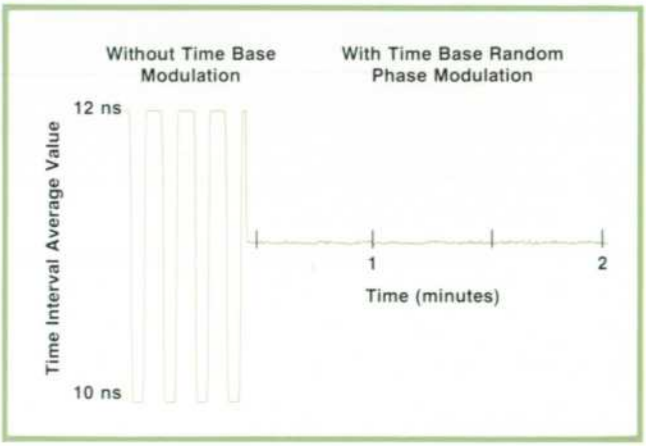

Fig. 3. *Effectiveness of time base random phase modulation is demonstrated by this time record of the counter reading during a time interval average measurement. Time intervals arrive at a rate of 50 MHz + 0.1 Hz, which is nearly coherent with the 5345A clock frequency of 500 MHz. Without modulation, the reading is either 10 ns or 12 ns. Modulation results in a true reading of approximately 11 ns.*

modulation (PM) of the clock is superior to frequency modulation (FM) for time interval averaging. First of all, it is the phase variations that destroy the coherence. Of course, phase variations are also generated by FM, but they decrease at 6 dB/octave as the modulating frequency is increased. Therefore, to obtain sizeable and rapid phase variations, relatively large FM signals must be used.

A more fundamental reason is that with FM, the mean value of the modulating function must be *absolutely zero* or errors will accumulate as the time base drifts. Zero mean modulation is difficult to obtain with nonlinear modulation circuits. With phase modulation, this zero mean value is not a necessity, because any constant phase offset is effectively cancelled by the start/stop process.

An Experiment

A simple experiment was performed to illustrate the effectiveness of phase modulation of the clock. Time intervals arriving at a rate nearly coherent with the 5345A clock frequency were measured with and without modulation. Fig. 3 shows the result.

Acknowledgments

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References

1. K. Jochim and R. Schmidhauser, "Timer/Counter/DVM: A Synergistic Prodigy?" and R. Schmidhauser, "Measuring Nanosecond Time Intervals by Averaging", Hewlett-Packard Journal, April 1970.

APPENDIX

Time Interval Estimation in the Presence of Quantization Error

For measuring short intervals by time interval averaging, only ±1 count quantization error is present. An interesting problem is to determine the statistics of estimating the true time interval from the reading obtained from averaging N independent measurements. The usual rule-of-thumb estimate gives this uncertainty as $T_0/\sqrt{N}$ where T_0 is the clock period. A more formal analysis shows that the actual rms uncertainty of the estimate is not so simply stated. In fact, the actual probability density function of the interval can be computed given the counter reading after averaging N independent measurements. The results can be summarized as follows.

An unknown time interval τ is measured N independent times with a clock period of T_0. The average value of the measurements is $\bar{\tau} = T_0(P+K/N)$ where P and K are integers and $0 \le K < N$. What is the probability density function of $e = (\tau - \bar{\tau})$ given P, K, and N? Assuming no *a priori* knowledge of τ, maximum likelihood estimation may be used, and the result is as follows:

Case I: K = 1,2,3,...,N−1. The probability density function P(x), that is, the probability density of $(\tau - \bar{\tau})$ taking on value x, is

$$P(x) = \frac{N+1}{T_0}\binom{N}{K}\left(1 - \frac{K}{N} - \frac{x}{T_0}\right)^{N-K}\left(\frac{x}{T_0} + \frac{K}{N}\right)^{K} \quad (1)$$

$$\text{for} \quad -KT_0/N \le x < (1-K/N)T_0$$

$$= 0 \quad \text{otherwise}$$

The mean of this distribution is $\frac{T_0(N-2K)}{N(N+2)}$ and the standard deviation is σ_τ, where

$$\sigma_\tau = \frac{T_0}{N+2}\left[\frac{(N-K+1)(K+1)}{N+3}\right]^{1/2} \quad (2)$$

Case II: K = 0. The probability density function P(x) for this case is

$$P(x) = \frac{N+1}{2T_0}\left(1 - \frac{x}{T_0}\right)^{N} \quad \text{for} \quad 0 \le x \le T_0 \quad (3a)$$

and

$$P(x) = \frac{N+1}{2T_0}\left(1 + \frac{x}{T_0}\right)^{N} \quad \text{for} \quad -T_0 \le x \le 0 \quad (3b)$$

The standard deviation of this function is σ_τ, where

$$\sigma_\tau = \left[\frac{2}{(N+2)(N+3)}\right]^{1/2} T_0 \quad (4)$$

These results show that the rms uncertainty in estimating the time interval by the average measurement value is a function of K as well as of N. The parameter K, always an integer, can be obtained directly from the measurement: K/N is the fractional part of the normalized measurement value $(\bar{\tau}/T_0)$. For large N, which is typical for most time interval average measurements, the rms uncertainty of equation 2 is reduced to

$$\sigma_\tau = T_0\left[\frac{\left(1 - \frac{K}{N}\right)\left(\frac{K}{N}\right)}{N}\right]^{1/2} \quad (5)$$

For a given N, the largest σ_τ, representing the worst case estimation uncertainty occurs when K/N is one-half. The corresponding rms uncertainty is $T_0/2\sqrt{N}$. The uncertainty becomes less when K/N approaches either 0 or 1. The rule-of-thumb estimate of $T_0/\sqrt{N}$, therefore, represents twice the worst case rms uncertainty for large N. One can further use the knowledge of the fractional part of the measurement value to increase confidence in estimating the time interval.

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Dave Chu holds BS, MS, and PhD degrees in electrical engineering, the BS from the University of California at Berkeley in 1961 and the MS and PhD from Stanford University in 1962 and 1973. At HP since 1962 (except for a 1967-69 leave of absence to teach physics and mathematics at Cuttington College in Liberia), Dave has contributed to the design of many of HP's top-of-the-line counters. A member of IEEE and the Optical Society of America, he's an expert on computer holography and inventor of the ROACH, a single transparency with controlled complex transmittance. He's authored many papers and patent applications in the fields of optics, information theory, statistics, and electronics. Dave was born in Hong Kong. He's married, has two children, and lives in a "shack"—which he's currently remodeling—on five acres of land in rural Woodside, California. He's active in his church and political party, and confesses to being addicted to Balkan and Israeli folkdancing.